Perturbation Analyticity and Axiomatic Analyticity. II. Hankel Transforms*

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(Received 10 January 1964; revised manuscript received 23 March 1964)

For a class of Feynman graphs $\mathcal{G}_n$, the perturbation analyticity over the external variables is studied with the aid of Hankel transformations over the internal mass parameters. Formal connection with the axiomatic analyticity is thereby established.

1. INTRODUCTION

IN paper I (preceding article) of the present series of study on the connection between the perturbation analyticity and the axiomatic analyticity, we have considered a class of Feynman graph $\{\mathcal{G}_n\}$.[1] It was shown there that the Landau singularity manifold $\mathcal{L}_n(z;\hat{a})$ of $\mathcal{G}_n$ can be parameterized in a simple matrix form which readily makes transparent a comparison with the known results in the axiomatic analyticity. Furthermore, it was pointed out that the boundary to such Landau singularity manifold is, in fact, formally equivalent to the p-space $Z=-\mathrm{D}\tilde{\mathrm{A}}\mathrm{N}\tilde{\mathrm{A}}\mathrm{D}$ form called (DANAD)′. This (DANAD)′ manifold, apart from the difference in the signs of certain parameters, has essentially the same structure as the x-space DANAD.

In the present paper, we wish to adopt an alternative approach in the establishment of the formal connection between the perturbation analyticity and the axiomatic analyticity. The procedures involved here are as follows:

(a) Consider the Hankel transform I_n of the Feynman amplitude H_n for the graph $\mathcal{G}_n$.

(b) With a set of off-diagonal masses[2] off the mass shells (in particular, $m_{kl}^2<0$), this distorted $\tilde{I}_n$ is formally identified with the Δ_n^+ function associated with the axiomatic primitive analyticity [Eq. (16) below].

(c) Thus, the distorted $\tilde{I}_n$ has as its singularity manifold the $\Xi_n(t)$ manifold.

(d) We then undo the distortion in $m_{kl}^2<0$ in $\tilde{I}_n$, i.e., continue analytically to the physical $m_{kl}^2>0$ in $\mathcal{G}_n$.

(e) Thus, the I_n has as its singularity manifold the $\Xi_n'(t)$ manifold.

(f) The boundary of perturbation singularity manifold is then the (DANAD)′ manifold.

This result coincides with that in paper I which was derived by a purely algebraic method.

2. HANKEL TRANSFORM OF THE p-SPACE PERTURBATION FUNCTIONS

The reason for doing this is as follows: Previously, there have been known examples[3,4] where one considers the x-space perturbation function in the ϕ^n coupling (Ward theory). It turns out much more convenient to consider instead the expression resulting from an integration over all mass parameters m_{ij}^2 with a certain weight function chosen as products of $\bar{\Delta}(m_{ij}^2,a_{ij})$. This operation has been called Hankel transforms, since the specific weight function used here is essentially a Bessel function[5]:

$$\bar{\Delta}(a,b)=\frac{1}{4\pi}\left[\delta(a)-\frac{b}{2}\theta(ab)\frac{J_1(ab)^{1/2}}{(ab)^{1/2}}\right]. \tag{1}$$

Thus, the Hankel transform of the x-space perturbation function turns out to be a different boundary value of the same analytic function that one gets from the Fourier transform of the time-ordered products, namely, the p-space perturbation function corresponding to a certain Feynman graph. In this way, a knowledge of the x-space perturbation singularity manifolds (in the space of external variables z_{kl}) is, in principle, obtainable from the knowledge of the corresponding singularity manifolds of certain Feynman graphs.

Hitherto, this formal manipulation proved fruitful only for $n\leqslant 3$, since there the p-space perturbation function can be easily handled and its singularity as well as the boundary surfaces determined in a straightforward manner. However, for $n\geqslant 4$ this is no longer so.

The possibility naturally suggests itself to a Hankel transform on the p-space perturbation function, if we are to look for a connection (within signs of certain parameters) between the p-space perturbation domain and the x-space axiomatic domain. Indeed, this inverse manipulation turns out tractable, especially since our knowledge of the axiomatic primitive domain is clearly in a much better shape, namely, we have both the DANAD and the $\Xi_n(t)$ manifold to rely upon.

We thus proceed to discuss in place of the p-space perturbation function,

$$H_n(q_k\cdot q_l;m_{ij}^2)=\int\cdots\int\prod_{i<j}dp_{ij} \times\frac{1}{p_{ij}^2+m_{ij}^2}\prod_{k=1}^{n-1}\delta(q_k-\sum_{i\neq k}p_{ik}), \tag{2}$$

* Supported in part by the U. S. Office of Naval Research.

[1] A. C. T. Wu, preceding paper, hereinafter quoted as I. $\mathcal{G}_n$ is the n-point loop diagram *with all internal diagonals*.

[2] In paper I, we pick a set of internal lines all connected to a given vertex. We call the masses associated with these lines the diagonal masses [cf. I, Eqs. (24), (30), and (32)]. The remaining $m_{kl}^2(k\neq l\neq 0)$ not connected with that particular vertex 0 are called the off-diagonal masses.

[3] G. Källén and A. S. Wightman, Kgl. Danske Videnskab. Selskab, Mat. Fys. Skrifter **1**, No. 6 (1958); especially Appendix III.

[4] A. C. T. Wu, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **33**, No. 3 (1961).

[5] G. Källén and J. S. Toll, Helv. Phys. Acta **33**, 753 (1960).

the Hankel transform of H_n. Again, to be specific, we take $n=4$; the extension to $n=5$ is obvious. We have

$$I_4(z_{kl};b_{ij})=\int\cdots\int\prod dm_{ik}^2 H_4(z_{kl};m_{ij}^2)\times\prod\bar{\Delta}(m_{ij}^2,b_{ij}). \quad (3)$$

As each propagator $(p_{ij}^2+m_{ij}^2)^{-1}$ is being folded by $\bar{\Delta}(m_{ij}^2,b_{ij})$, we note that a conventional $i\epsilon$ prescription in the form of $(p^2+m^2-i\epsilon)^{-1}$ would give back the Δ_F functions. However, it is well known that the two-point functions Δ_F and Δ_2^+ (apart from trivial normalization constants) are different boundary values of the *same* analytic functions. Therefore, out in the complex $-p^2$ plane, we have[5]

$$\Delta_2^+(-p^2;b)=\frac{1}{\pi i}\int_0^\infty dm^2\frac{\bar{\Delta}(m^2;b)}{p^2+m^2}. \quad (4)$$

As the structure of the multiparticle Green's functions (3) is essentially governed by the conglomeration of the 2-point functions,[6] the $i\epsilon$ shift in the denominators of (3) will not alter the *forms* of the singularity manifolds of the n-point functions when the totality of the Riemann surfaces are taken into account. The question of determining which portions of the singularity manifolds actually belong to the physical sheet is a question which should properly belong in a discussion of the relevance criteria of the boundary. Since our primary purpose at this stage is just to write down the whole singularity manifolds and to determine the forms of the boundaries, we postpone all discussions on the relevance criteria.

We now get from (3) and (4)

$$I_4(z_{kl};b_{ij})=(\pi i)^6\int\cdots\int\prod dp_{ij}\prod\Delta_2^+(-p_{ij}^2;b_{ij})\times\prod_k\delta(q_k-\sum_i p_{ik})$$

$$=\frac{1}{64(2\pi)^{12}}\int\cdots\int\prod_{k=1}^{3}dy_{0k}\exp(i\sum_l q_l y_{0l})\times\prod_k\theta(y_{0k})\delta(y_{0k}^2+b_{0k})\prod_{k\neq l\neq 0}dy_{kl}\theta(y_{kl})\times\delta(y_{kl}^2+b_{kl})\delta(y_{kl}+y_{0k}-y_{0l}). \quad (5)$$

The integrals over $y_{kl}(k\neq l\neq 0)$ are trivially carried out with the "triangular" relations[7]

$$y_{kl}=y_{0l}-y_{0k}\quad k\neq l=1,\cdots,(n-1); \quad (6)$$

and all the δ functions can be written as

$$\delta(y_k y_j+a_{kj}'), \quad (7)$$

where

$$y_k\equiv y_{0k}, \quad (8)$$

$$a_{kk}'=b_{0k}\quad k=1,2,3, \quad (9)$$

$$a_{kl}'=-\tfrac{1}{2}(b_{kl}-b_{0k}-b_{0l})\quad k\neq l\neq 0. \quad (10)$$

Equation (5) simply reads

$$I_4(z_{kl};a_{kl}')=\frac{1}{64(2\pi)^{12}}\int\cdots\int\prod_k dy_k\exp(i\sum_l q_l y_l)\times\prod_k\theta(y_k)\prod_{k\leq l}\delta(y_k y_l+a_{kl}'). \quad (11)$$

Before identifying it with Δ_4^+, it is necessary to ascertain the range of the "mass" parameters a_{kl}' in (11). From the explicit expression of $\bar{\Delta}(m_{ij}^2;b_{ij})$ given in (1), it is clear that the signs of b_{ij} should be the same as the signs of m_{ij}^2. For physical masses in the graph $\mathcal{G}_4$, $m_{ij}^2\geqslant 0$ implies that $b_{ij}\geqslant 0$. Then (9) gives

$$a_{kk}'\geqslant 0. \quad (12)$$

However, the signs of the nondiagonal a_{kl}' in (10) are not necessarily positive semidefinite, especially when b_{0k} become arbitrarily small. To circumvent this, we first formally let $m_{kl}^2(k\neq l\neq 0)$ go off the mass shell and take on negative values. We shall call the resulting function the *distorted* p-space perturbation function and add a tilde sign to all affected quantities. Call $b_{kl}=-\tilde{b}_{kl}$, and we have

$$\tilde{b}_{kl}>0,\quad (k\neq l\neq 0). \quad (13)$$

Equation (13) then guarantees that

$$\tilde{a}_{kl}'>0,\quad (k\neq l\neq 0). \quad (14)$$

With (12) and (14), it is now possible to define a region which is the intersection of the following.

$$(-1)^r\det_{(\mathrm{rank}\ r)}|\tilde{a}_{kl}'|\leqslant 0,\quad 1\leqslant r\leqslant 3. \quad (15)$$

Therefore, by definition [see I, Eq. (9)], the right-hand side of (11) is indeed a Δ_4^+-function, or

$$I_4(z_{kl};\tilde{a}_{kl}')=\frac{-i}{64(2\pi)^3}\Delta_4^+(z_{kl};\tilde{a}_{kl}'). \quad (16)$$

This shows that the Hankel transform of the distorted p-space perturbation function has formally the same structure as the generalized singular function $\Delta_4^+(z_{kl},\tilde{a}_{kl}')$, the singularity manifold of the latter being the $\Xi_4(t)$ manifold defined in I, Eq. (11).[8,9] The boundary

[6] A very instructive example is furnished by the addition theorem (Sonnie's formula) on the Bessel functions. See, e.g., G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, New York, 1944), 2nd ed., p. 367. This was first applied to Δ_n^+ in the work of Källén and Wilhelmsson, Ref. 8. It shows specifically how the higher order complex singularity, so to speak, may be generated from cut planes.

[7] Equation (6) is indeed similar to the triangular loop equations we encounter for the Feynman graph $\mathcal{G}_n$, cf. especially I, Eq. (20). Thus, y_{kl} here is to be contrasted with $\alpha_{kl}p_{kl}$ there.

[8] G. Källén and H. Wilhelmsson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Skrifter **1**, No. 9 (1959).

[9] G. Källén, Lectures Notes (Les Houches, 1960) in *Relation de dispersion et particules élémentaires* (Hermann & Cie, Paris, 1960), p. 389.

to this singularity manifold is then given when the diagonal elements $a_{kk}'=0$ and the $\frac{1}{2}(m-1)(m-2)$-mass envelope is taken over the remaining nondiagonal elements $\tilde{a}_{kl}'>0$.[10]

The connection with the Landau singularity manifold with physical internal masses lies then in redefining a distorted $\Xi_n(t)$ manifold, called the $\Xi_n'(t)$ manifold, which is now parameterized by t, $(n-1)$ diagonal $a_{kk}\geqslant 0$, and $\frac{1}{2}(n-1)(n-2)$ nondiagonal $a_{kl}\leqslant 0$. For $n=4$, the $\frac{1}{2}(n-1)(n-2)$-mass envelope of this $\Xi_4'(t)$ manifold gives the (DANAD)$'$ manifold discussed in paper I. Note that the right-hand side of (16) has a one-parameter integral representation[8] with the explicit denominator $[\Xi_4(t)]^{1/2}$. It is therefore possible to continue analytically both sides to the physical masses.

To what extent it is possible to recover the analyticity properties of the Feynman amplitude $H_n(z;m^2)$ itself from those of the Hankel transform I_n remains to be discussed. As far as the analyticity properties in the external momentum variables z_{kl} of the p-space perturbation functions are concerned, we could very well consider functions which are superpositions of graphs $\mathcal{G}_n$ (for a given n) with arbitrary internal masses, i.e., integral of H_n over internal masses with arbitrary weight functions. The *maximal* singularity manifolds in z that are, in principle, allowed for the p-space perturbation functions with such superpositions will then be the Landau singularity manifolds $\mathcal{L}_n(z;\hat{a})$ discussed in I, when considered over the totality of the Riemann surfaces.[11]

[10] G. Källén, Nuclear Phys. **25**, 568 (1960).

ACKNOWLEDGMENT

The author is greatly indebted to Professor G. Källén for his interest and encouragement.

[11] A similar situation holds in x space: The maximal singularity manifolds for the vacuum expectation values is given by those from the Δ_n^+ functions.

PHYSICAL REVIEW VOLUME 135, NUMBER 1B 13 JULY 1964

π^--p Interactions at 604 MeV*†

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(Received 2 August 1963; revised manuscript received 5 March 1964)

The interactions of 604 MeV π^- mesons in a hydrogen bubble chamber have been systematically analyzed. In 33 000 pictures a total of 8052 usable events were found, corresponding to cross sections of 18.9±1.3 mb for σ(elastic), 4.98±0.54 mb for $\sigma(\pi^- p\pi^0)$, 7.87±0.91 mb for $\sigma(\pi^- n\pi^+)$, 14.0±1.0 mb for σ(neutrals), with σ(two-pion production) <0.2 mb, for a total cross section of 45.9±1.9 mb at this energy. The angular distribution for elastic scattering was fitted with a fifth-order polynomial in cosθ which gave a value of $d\sigma/d\Omega(0°)$ consistent with dispersion theory. The pion-pion effective-mass distributions for both single-pion-production channels showed pronounced peaking at high mass values, strongly inconsistent with simple isobar-production kinematics. Simple one-pion exchange does not appear to play a significant role.

I. INTRODUCTION

PION production above the ρ threshold is known to be dominated by reactions in which the pion resonance states are excited.[1–5] At lower energies it might be expected that most of the interaction would proceed through isobar-production channels,[6] with some possible $T=0$ effect due to the ABC phenomenon.[7] This paper reports evidence of peaking in the effective-mass distributions of both pion-production channels which is inconsistent with the conventional isobar picture, and which suggests that the reaction mechanism is considerably more complicated than has heretofore been expected.[8]

* Supported in part by the National Science Foundation.

† A portion of this work has been submitted by C. N. Vittitoe in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the University of Kentucky.

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[7] N. E. Booth, A. Abashian, and K. M. Crowe, Phys. Rev. Letters **7**, 35 (1961).

[8] See, for example, the rapporteur talk of G. Puppi, *Proceedings of the 1962 International Conference on High Energy Physics at* CERN (CERN Scientific Information Service, Geneva, 1962), p. 722.